ON THE INTRINSIC STRUCTURE OF CALCULUS

By

Andrew Grossfield
College of Aeronautics

Abstract

This article contributes to the calculus reform movement by presenting a topic structure for calculus. A tree structure can be used to convey an overview of the topics treated in calculus/college algebra courses. The tree diagram is made available and a set of lists containing the kinds, properties, operations, and forms for the objects of calculus, specifically, functions. The tree structure will assist faculty in preparing lectures, and provide students with an organizational study aid as well as guide authors preparing texts. The idea of structuring topics applies not only to mathematics courses but is well suited to the studies of mechanics, electronics and other technical areas.

Background

Peter D. Lax in a recent address[2] said:

> There is, at long last, a consensus that the teaching of calculus has been asleep for fifty years. ...the sleeper is beginning to stir.
>
> During this long slumber, a tremendous amount of detritus has accumulated in the standard calculus course. Vestigial remains and rococo excrescences obscure its main ideas and applications, ... The first and most urgent task is to ruthlessly remove the weeds.

It's not just some weeds that need to be removed. The whole garden needs landscaping. To begin, ask the basic questions. What is being studied here and why?

The answers are that the major objects of study in calculus/college algebra are relationships between variables, in particular, functions. The reason is that the major concerns in every quantitative scientific area are the relationships between the variables of that area. Since in the classical quantitative areas these relationships were continuous and smooth, the mathematicians of yesteryear developed the concept of the function and the techniques of using functions. The concept has proven to be a success and deserves a place of honor in the study of mathematics along with numbers and shapes. Technical competency in our modern world requires a thorough understanding of functions.

Pedagogical Problems

College algebra, where many students first see functions, is a hodgepodge. Topics include discriminants, determinants, synthetic division, logarithms, the laws of sines and cosines, sequences, probability and statistics and more! Amid all the details of college algebra and calculus many students never acquire the comprehension that something called a function is being studied.

The subject is distributed throughout a 500 page text while the student can view only two pages at a time. Concepts are introduced, named and dropped, it appears, for no apparent reason. Techniques of form changing are stressed without examining the forms themselves or their supporting concepts. What sense can it all make? Is calculus presented differently?

The Tree Structure

One cannot blame students, faced with the many kinds, forms, properties, operations, and combinations of functions for being unable to keep track of or comprehend all the details. The textbook format requires a linear layout of concepts and ideas. The mind appears to absorb information linearly. However, all the concepts exist, at the same time, as part of the acquired knowledge of humankind.

Computer programmers convey information by means of a tree structure, a screenful at a time. With all the publicity devoted to the benefits of structured programming, it is time for us, as faculty, to devote a similar effort to structuring the content of college algebra and calculus. The structure should determine the placement of topics.

Calculus is the study of continuous, smooth functions. Restricting calculus to mean only that part of the study of functions pertaining to limits, derivatives and integrals fragments the structure and sacrifices clarity.

A reasonable introduction to any technical subject would first present a description of the objects of study, and then descriptions of kinds, properties, operations and forms. The tree structure for the study of functions, would place functions at the root, with kinds, properties, operations, forms and objectives in the first sub-level (See Figure 1). Perhaps the best name for the course is Functions. Alternate names such as Calculus, Advanced Algebra, College Algebra, and Precalculus, lack focus and do not describe the subject to a beginner.

Functions

Our subject should begin with a description of functions; what they are, and why they pervade so many areas of quantitative science. The concept is neither obvious nor simple. Explanation is required.

The essential idea of a function is neither “ordered pair-ness” nor “singlevalued-ness.” It is relationship between variables, perhaps, control. A student who does not know what a function is or why he is studying functions is wasting his time memorizing details in a calculus class. A conscientious faculty member will ascertain that every student is aware of the importance of functions and will relate every lecture to that major concept. Students who are not convinced that functions are real and have immutable properties, will have difficulty in comprehending or mastering any of the detailed aspects of functions or techniques of applying functions?

Kinds

A breakdown (see Figure 2) of the kinds of functions depicts a classification based on the operations involving the independent variable. The five classes include all the functions that are treated in traditional calculus courses. My previous article[1] provides a description of the allowable graphical behavior permitted to members of the different classes of functions. A technician, in choosing a good functional model for some process that is under study, must be conscious of the properties of the functions that describe the phenomena under study. Regrettably, in too many texts, the various kinds of functions are not contrasted. What is more natural to human beings than to distinguish, name and sort objects?

Properties

Conventionally, the properties of curves (see Figure 3) have been segregated by the testing or evaluation techniques required by the property. A student may benefit by a discussion of curves with all their properties treated equally. How marvelous that differential calculus can predict behavior such as monotonicity, extrema, concavity and points of inflection. Many of the properties of curves have simple graphical interpretations.

Forms

Some students ask, what is the difference between a kind and a form. It is a wonderful question. Forms (see Figure 4) represent a major mathematical idea that is poorly treated in most math courses. Only techniques of form changing are stressed. Topics, such as: why are there different forms, why should we change forms, which is the best form and why don't we use that form alone, and what is the connection between forms and identities are rarely addressed. Do polynomials have identities? Placing forms in the first sub-level of the tree structure recognizes the major importance of the concept and provides structure for many of the concepts such as identities which seem to jump out of nowhere. The strategic relationship between symbolic and graphical forms can not be over stressed.

Operations

The list of operations also reveals inconsistencies in Calculus texts. Again, while graphical integration and differentiation are stressed, students may not confront graphical addition, subtraction, multiplication or division. It appears that faculty are assuming these topics are either unimportant or are treated in some other course.

Objectives

Another development is a list (see Figure 6) of problems that a calculus student will be able, using techniques of calculus, to solve by the end of the course. The list is limited to objectives involving only functions and curves. There is no point in explaining something that a student does not understand in terms of something else which the student also does not understand. Many other applications such as electronics or thermodynamics are better treated in courses specifically devoted to those applications.

Summary

The structure directs students to the most important course concepts. Provided with a strategic framework, a student can choose his own order in which to study the course topics thereby acquiring some control over his study. Provided with a proper framework, a student can see why each kind, form and operation may require its own rule for differentiation and integration. Parsing of expressions for the purposes of integration or differentiation will be more meaningful. Provided with a proper framework, a student can see the end of the course from the beginning. Reproduce the tree structure, figure 1, and give it to the students from the very beginning. National policy should ensure grade schoolers structure their study of numbers and shapes.

Indeed, the new software and calculators remove much of the drudgery formerly encountered by students and technicians in solving complex problems. Although calculators and software will change, the concepts, ideas and structure of calculus will endure. Lest we forget our role, faculty will still be required to convey the judgment remaining in formulating a problem, in selecting appropriate kinds and forms of models, in choosing methods of solution and in interpreting results.

Acknowledgments:

The author would like to thank Cynthia Z. Barksdale, Rodney Dash and David Nixon for their help in preparing the manuscript.

References:

1. Grossfield, Andrew "On the Classification of Functions and Curve Plotting" Proceedings of the 1990 ASEE Annual Conference, Session 2665 (1782-1784)

2. Lax, Peter D. Comments on Calculus Reform. Focus on Calculus. Issue No. 8. New York: John Wiley & Sons, Inc.

Forms of Functions

Tables			
Graphs	rectangular		
	polar		
Symbolic	direct		
	inverse		
	implicit		
	composite (chain)		
	parametric		
	polar		
	series	taylor	
		fourier	

Figure 4

Kinds of Functions

polynomials
rational functions
algebraic functions
transcendental functions
piece wise defined functions

Figure 2

Properties of Functions and Curves

(Distinguish point properties from region properties)

1. intercepts,
2. symmetries, even, odd, periodic, etc.
3. excluded intervals,
4. bounds,
5. asymptotes,
6. multivalued,
7. continuity,
8. discontinuity
    - isolated discontinuities
        - finite jumps
        - infinite jumps, poles
        - gaps
        - oscillatory discontinuities
9. differentiability,
10. slopes,
11. monotonicity,
12. extrema,
13. concavity,
14. points of inflection,
15. cusps,
16. radii of curvature,
17. arc length,
18. area
19. connected
20. closed,

Figure 3

Operations on Functions

unary operations: af(x) , -f(x) , $\frac{1}{f(x)}$,

$(f(x))^2$, $(f(x))^n$, $\sqrt[n]{f(x)}$,

$\frac{df(x)}{dx}$, and $\int f(x)dx$,

binary operations: + , - , * , /

and $(f(x))^{g(x)}$

Figure 5

Objectives of Calculus

Evaluation:

- evaluating functions of a single variable
- graphing functions
- finding roots
- finding intersections
- evaluating inverse functions

Differentiation:

- finding extrema and points of inflection
- finding tangent lines
- describing direction
- describing rate of change

Integration:

- finding areas of regions with curved boundaries
- finding arc length of curves
- finding surface area and volumes of revolution

Figure 6.

Algebra/Calculus Topic Tree Diagram

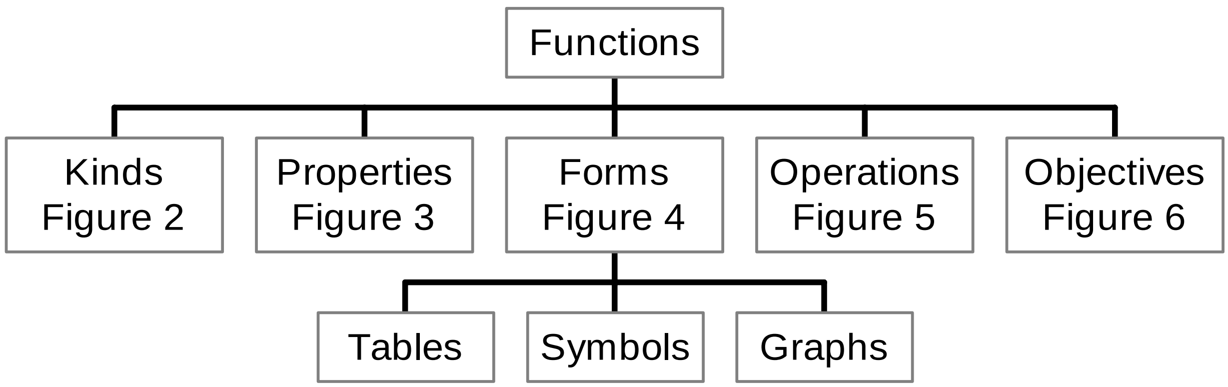

Figure 1

ANDREW GROSSFIELD

Throughout his career, Dr. Grossfield has combined an interest in engineering design and mathematics. He earned his BSEE at the City College of New York. During the early sixties, he obtained an M.S. degree in mathematics at night while working full time during the day, designing circuitry for aerospace/avionics companies. As a Graduate Associate, pursuing a doctoral degree at the University of Arizona, he found himself in the odd position of both teaching calculus courses and taking courses in applied mathematics. Being caught in the middle made him acutely aware of the differences in mathematics as seen by the mathematician, as needed and used by the engineer, and as presented to the student. He is licensed in New York as a Professional Engineer and is a member of ASEE, IEEE, and SIAM.