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Freedom, Constraint and Control in Multivariable Calculus

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Abstract

Certainly, everyone interested in technology should possess an understanding of the models of deterministic, continuous multivariable control. The study of multivariable calculus can be viewed as a natural extension of the unfortunately named "single variable" calculus. Ordinary "single variable" calculus is the study of equations in two variables, $F(x, y) = 0$. Equations in two variables provide mechanisms for studying:

a) continuous curves in the two dimensional plane,
b) continuous models of control (where one variable controls a second variable),
c) continuous models of time variation (signals, trends or evolution) or
d) situations where two variables track together continuously.

These functions and curves of ordinary calculus are real entities having observable and predictable properties.

What happens when a single variable depends on or is controlled by several other variables? How do we visualize and treat the situation when several variables depend on or are controlled by a single variable? What if several variables control several other variables? How will small changes in the controlling variables affect the controlled variables?

This paper begins with a discussion of the importance of counting variables at the beginning of multivariable problems, proceeds to a classification of the entities of multivariable calculus and continues to describe the concept of the derivative as it pertains to each of the entities of multivariable calculus.

Similar to the functions of calculus, the objects of multivariable calculus, including space curves, warped surfaces and models of multivariable control are also real and also have observable and predictable properties. Every additional variable provides an additional degree of freedom by adding a dimension to the space spanned by the collection of variables. Every additional equation provides a constraint, which removes a dimension from the space spanned by the variables.

Counting Variables

A study of the numeric properties of rectangles might begin by listing the important quantitative properties of rectangles and noting relationships, which hold among these properties. These quantitative properties would be listed as the variables:

L	the length of a rectangle,
W	the width of the rectangle,
A	the area of the rectangle,
P	the perimeter of the rectangle and
D	the diagonal of the rectangle.

The values for some of these variables can be arbitrarily specified but not all of the variables can be chosen arbitrarily. What are the rules that determine the available possibilities?

Relationships between the variables, which are true for all rectangles must hold. In this case these relationships are the equations:

$$A = L*W,$$
$$P = 2(L + W) \text{ and}$$
$$D^2 = L^2 + W^2.$$

Each equation permits one variable to be evaluated and therefore the three equations permit three of the variables to be evaluated. In this situation any two variables can be considered as independent and if properly chosen, the remaining three variables will be determined. If n is the total number of variables and m, the number of equations, the number of independent, free variables is n – m. The following cases should be recognized:

A. If the number of equations equals the number of variables the solution might consist of a point or maybe several isolated points.

B. If the number, n, of variables is one more than the number of equations, the solution will be a curve that is, a one-dimensional manifold, in n-space.

In three-dimensional space two canonical forms for a curve should be recognized:

i. two equations, each in three variables: each equation represents a surface and the curve is the intersection of the two surfaces:

$$F(x, y, z) = 0$$
$$G(x, y, z) = 0$$

ii. three equations, each displaying how a point, P(x, y, z) is controlled by a parameter:

$$x = f(u)$$
$$y = g(u)$$
$$z = h(u)$$

Here if u represents time, t, the three equations describe how the point, P(x, y, z) moves with time. The parameter, u, sometimes is chosen to represent the distance s along the curve from a fixed point $P_0(x_0, y_0, z_0)$ to the moving point P(x, y, z) .

C. If the number, n, of variables is two more than the number of equations the solution will be a surface that is, a two-dimensional manifold, in n space, etc.

Here too, in three-dimensional space two canonical forms for the surface should be recognized:

i. $F(x, y, z) = 0$, x and y determine z.

ii.
$$x = f(u, v)$$
$$y = g(u, v)$$
$$z = h(u, v)$$

In case ii, u and v provide co-ordinatizations for the surface, each choice of the parameters u and v determines the location of a point P(x, y, z) on the surface.

D. If the number, n, of variables exceeds the number of equations by three we might try to visualize the resulting manifold as a one parameter family of surfaces or if the parameter is time, we may see it as a surface moving as a function of time.

In what follows the equations will be assumed to be independent and if the number of variables is limited to three, three-dimensional spatial visualization can be applied.

Families

A moving surface in 3-space can be described by an equation in the 3-space variables and a parameter, $F(x, y, z, p) = 0$. For each fixed value of the parameter, p, a surface is determined. The entirety of such surfaces is called a family of surfaces. Upon examining the parametric form of a surface in 3-space,

$$x = f(u, v)$$
$$y = g(u, v)$$
$$z = h(u, v)$$

it is seen that when v is specified, a curve with parameter u is obtained. If u is chosen, a different curve with parameter v is obtained. The surface is crosshatched with two curve families. Every point on the surface lies at the intersection of one member of each of the families. The parameters, u and v provide a set of coordinates for the points on the surface. Consider the sphere, of radius R. The sphere can be represented by the set of equations:

$$x = R \cos(\phi) \cos(\theta)$$
$$y = R \cos(\phi) \sin(\theta)$$
$$z = R \sin(\phi),$$

where ϕ and θ represent the spherical surface co-ordinates, latitude and longitude. Distinct, constant values of θ determine distinct great semi-circles of longitude; while distinct constant values of ϕ generate the small circles of latitude lying in planes parallel to the plane of the equator.

Non-linear Spatial Co-ordinate Systems

To provide a co-ordinate system for a 3-dimensional space we must cover the space with three intersecting families of surfaces:

$$F(x, y, z, p) = 0 \quad ,$$
$$G(x, y, z, q) = 0 \quad ,$$
$$H(x, y, z, r) = 0 \quad .$$

Then to each choice of p, q, and r we can assign a point P(x, y, z)which is the intersection of the three surfaces corresponding to constant p, q and r. The parameters, p, q, and r can serve as co-ordinates for the points of our three-dimensional space. As an example with cylindrical spatial co-ordinates the families of surfaces with co-ordinates r, θ , z are:

$\sqrt{(x^2 + y^2)} = r$	circular cylinders concentric with the z-axis, expanding as r increases,
$\arctan(y/x) = \theta$	pencil of planes rotating through the z-axis as θ increases,
$z = z$	family of horizontal planes which rise with increasing z.

Vectors and n-tuples

Not all n-tuples are vectors. An n-tuple is a set of variables **(**u, v, w**)** which span a possibly non-linear space. For example, in chemistry the state of a gas is described by the variables pressure, specific volume and temperature. A point (p, v, T**)** in the space spanned by these variables is not a vector. There is no meaning in adding pressure to temperature. In the following, when n-tuples and vectors must be distinguished, n-tuples will be written with parentheses as **(**u, v, w). In contrast, column vectors, members of a linear vector space will be written with brackets as:

$$\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and row vectors as } \mathbf{b} = [a, b, c]\textbf{.}$$

Entities of Finite Dimensional (Multivariable) Control

Finite multivariable control refers to situations where m variables control n variables. These situations can be described by n equations, each in n + m variables. Selecting m variables determines the remaining n variables. Varying any of the m variables forces the remaining n variables to vary. A useful conceptual categorization of finite dimensional control includes four categories.

1) An ordinary calculus function, that is, one variable controlled by another (a curve in 2-space): $y = f(x)$. Here a scalar variable is controlling a scalar variable.

2) One variable (dependent) controlled by several (a surface in n-space), called a scalar field if the controlling variables are position coordinates:

$$w = f(x, y, z) \text{ or } w = f(\mathbf{r}). \qquad \text{In this case a 3-tuple is controlling a scalar.}$$

3) Several variables controlled by one (a curve in m-space):

$$\begin{aligned} x &= f(t) \\ y &= g(t) \qquad \text{or } \mathbf{r} = \mathbf{r}(t) \\ z &= h(t) \end{aligned}$$

Here a scalar variable is controlling a 3-tuple.

4) An n-tuple of n variables controlled by m variables, usually called a "transformation" or "mapping," $\overline{T}$, from m-space to n-space):

$$U = f(w, x, y, z)$$
$$V = g(w, x, y, z)$$

or in n-tuple notation $\overline{Y} = \overline{T}(\overline{X})$ where $\overline{X} = (w, x, y, z)$ and $\overline{Y} = (U, V)$.

If the m controlling variables represent spatial coordinates and the n controlled variables have vector properties, this entity is called a vector field. Because the words, transformation and mapping, are abstract and because perhaps the word function should be reserved for the ordinary calculus functions, category 1, with one independent and one dependent variable, the entities of the above categories 2, 3, and 4 will be called the multivariable models of control. In accordance with convention, the maybe non-linear space spanned by the controlling variables will be called the "domain" and the maybe non-linear space spanned by the controlled variables the "range."

Linear Spaces

Linear spaces are flat, contain the origin and have linear co-ordinate systems. In this context flat means not curved or warped. One-dimensional linear spaces must be straight lines through the origin. Two-dimensional linear spaces must be planes through the origin. Spheres, paraboloids, cones and helices are inherently not linear. Linear co-ordinate systems are characterized by co-ordinate lines or surfaces that are straight or flat, parallel and uniformly spaced. As an example, the family of straight lines through the origin, $y = mx$, cannot serve to co-ordinatize a linear plane. As another example, the family of circles concentric with the origin, $x^2 + y^2 = r^2$, cannot serve to co-ordinatize a linear plane. As a last example, co-ordinate lines that are logarithmically spaced, cannot serve to co-ordinatize a linear plane.

In linear vector spaces co-ordinatization of points is abandoned in favor of description by means of basis vectors. It can be shown that every choice of basis vectors can be associated with a linear co-ordinatization and vice versa. The points of a linear space can be represented by vectors whose components can be added in accordance with the laws of vector addition. Flat spaces that do not contain the origin but which possess linear co-ordinate systems are not linear spaces. The word affine is used to describe these spaces.

Tangent Spaces

The tangent manifold to an n-dimensional manifold is n-dimensional, linear, touches the manifold and has the same spatial orientation as the manifold does at the point of contact. The tangent line to a space curve touches the curve and has the same direction, as does the curve at the point of contact. The tangent plane to a surface touches the surface and has the same perpendicular direction, as the surface has at the point of contact. If the equations for a curve are:

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t),$$

then the equations for the tangent line at a point $P_0(t)$ on the curve will be:

$$dx = f'(t)\,dt, \qquad x - x_0 = dx$$
$$dy = g'(t)\,dt \qquad y - y_0 = dy$$
$$dz = h'(t)\,dt \qquad z - z_0 = dz$$

Here dx, dy, dz, and dt are ordinary variables which describe the deviation of a point $P_1(dt)$ on the tangent line from the point P_0. There is no need to consider the "differential" variables dx, dy etc. as infinitesimally small. The tangent line may have infinite extent and so may these variables. The scalar variable dt provides a linear co-ordinate system for the one-dimensional tangent line.

If the equation for a surface is: $f(x, y, z) = 0$, the equations for the tangent plane at the point $P_0(x, y, z)$ on the surface will be:

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial x} dz = 0$$

where again $dx = x - x_0$, $dy = y - y_0$, and $dz = z - z_0$ and the partial derivatives are constants at P_0.

Here dx, dy, and dz are variables which describe the deviation of a point $P_1(x, y, z)$ on the tangent plane from the point P_0. The variables dx and dy provide a linear co-ordinatization of the tangent plane to $f(x, y, z) = 0$ at P_0 . Again the co-ordinate "differential" variables, dx, dy, and dz may be unbounded in extent.

Linear Control Models

The linear multi-variable control entities are special cases of the models of the general non-linear multivariable control described above. The conditions required for the control to be linear are:

1) the space of the controlling variables must be linear,
2) the space of the controlled variables must be linear and
3) the equations relating the controlling and controlled variables must be first degree with no constant term, that is, linear.

With these conditions the forms for the four models of linear control become:

1) one scalar variable linearly controlling another: $y = mx$, described by a straight line through the origin in the two-dimensional Cartesian direct product space formed from the spaces of the controlled and controlling variables.

2) one scalar variable linearly controlled by many: $w = ax + by + cz$,

3) Several variables linearly controlled by one, (a straight line through the origin in n-space):

$$x = at \qquad y = bt \qquad z = ct \qquad \text{or in vector notation} \quad \overline{Y} = t\,\overline{N}.$$

where $\overline{Y}$ = [x, y, z] and the vector $\overline{N}$ = [a, b, c] represents the direction numbers of the line.

4) An m-tuple of m variables linearly controlled by n variables, usually called a "linear transformation" or "linear mapping" from n-space to m-space). Below the variables v, x, y, and z linearly control the variables p, q and r.

$$p = a_{11}v + a_{12}x + a_{13}y + a_{14}z$$

$$q = a_{21}v + a_{22}x + a_{23}y + a_{24}z$$

$$r = a_{31}v + a_{32}x + a_{33}y + a_{34}z$$

Linear control can be described by a matrix $[a_{ij}]$ where the value of each component a_{ij} represents the effect of the influence of the controlling variable x_j on the controlled variable y_i .

Linear control models, LCM's, possess the following properties:

- The dimension (the number of degrees of freedom) of the range cannot exceed the dimension of the domain even if there are more controlled variables than controlling variables.
- continuity, which means small changes in the values of the controlling variables produce small changes in the values of the controlled variables,
- LCM's preserve boundedness, (that is the images of bounded regions in the domain will be bounded in the range),
- LCM's preserve tangency, that is if two curves in the domain are tangent at a point P_0 their images in the range will be tangent at the image point of P_0,
- The image of an algebraic curve of degree p will be an algebraic curve of degree p. As a corollary, the images of straight lines are straight lines.
- The images of parallel lines in the domain will be parallel lines in the range.
- The images of uniformly spaced lines in the domain will be uniformly spaced in the range.

Therefore the image of a parallelepiped in the domain will be a parallelepiped in the range. And furthermore the image of an ellipse inscribed in a parallelogram in the domain will be an ellipse inscribed in a parallelogram in the range.

- If the range and the domain have the same dimension, say three, and the coordinate systems are ortho-normal, then the ratio of the parallelepiped volumes (range/domain) will equal the determinant of the control matrix, $[a_{ij}]$. Such matrices are called invertible or non-singular.
- The ratio (image/pre-image) of volumes of any shape will equal the determinant of the control matrix, $[a_{ij}]$ if the coordinate systems are ortho-normal.
- If the determinant of the control matrix, $[a_{ij}]$ is not zero then another matrix can be found which will allow the pre-image of any point in the range to be found uniquely.

- If the determinant of the control matrix is zero, then the dimension of the range is less than the dimension of the domain.

Direct Product Spaces and Visualization

The functions that are studied in ordinary calculus are viewed in what is called the direct product space. The space of the controlling variable, x, and the space of the controlled variable, y, each one-dimensional are combined into a two-dimensional x-y space in which the relationship can be viewed. It is a remarkable invention, this combination called the direct product space which permits the association of a curve with a functional relation. This invention is the foundation of Analytic Geometry.

Here are some examples of cases where the number of variables is small enough so that the relationship can be visualized. If variables, x and y, control variable z, two-dimensional x-y space can be combined with the one-dimensional z space to produce a surface in the direct product x-y-z space. If a variable t controls variables x and y, the x-y relationship can be plotted as a parametric form in the two dimensional x-y space. If a variable t controls variables x, y and z than perhaps one can imagine the curve as a parametric form in x-y-z space.

When the sum of the variables of the controlling and controlled spaces equals or exceeds 4, the direct product visualization technique may fail to be useful or possible. If two variables, x and y, are controlling two other variables, u and v, as in the theory of functions of complex variables, the two-dimensional x-y space is kept separate from the two-dimensional u-v space. To grasp the connection between points (x, y) in the space of the controlling variables and points (u, v) in the space of the controlled variables, we may try to view how a curve family in the x-y domain maps into a curve family in the u-v range. In general, finding a model that displays the important features of relationships in which many variables control many other variables may not be easy.

Derivatives of the Multivariable Control Models

Four prototypes of models of multivariable control exist. Each will have its own form of derivative. But first, a meaningful definition of derivative that is general enough to apply to each of the four prototypes should be formulated. Consider a non-linear mapping from the domain, D, the space of controlling variables to the range, R, the space of the controlled variables. Consider the tangent space, $T_{D,}$ co-ordinatized by the differentials of the controlling variables at a point, P_1 in the domain. Consider the tangent space, $T_{R,}$ at P_2 in the range of the image point of P_1 . Let the tangent space, T_R, be co-ordinatized by the differentials of the controlled variables. Define the multivariable derivative at P_1 as the linear mapping from T_D to T_R, each co-ordinatized by their respective differential variables.

Consider the information provided by the derivative of the ordinary functions of calculus and the information provided by the above the multivariable derivatives. The derivative of an ordinary function, $y = f(x)$ at a point x_0 provides a linear relationship between the differential variables dy and dx:

$$dy = f'(x_0)\, dx.$$

The derivative $f'(x_0)$ describes the direction of the curve $y = f(x)$ at the point $P(x_0, y_0)$in the x-y direct product space.

In case 2 where a single variable, w, depends on three variables x, y and z, the multi-variable derivative, called the gradient, ∇f, of the function, $w = f(x, y, z)$ provides a linear relationship between the differentials dx, dy and dz in the tangent space of the controlling variables and dw.

$$dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial z} dz = \nabla f \bullet \overline{dr}$$

where grad f,

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

acts on the differential position vector,

$$\overline{dr} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

For a surface in three space, $F(x, y, z) = 0$, grad F, at the point $P(x, y, z)$, provides the direction of the normal to the tangent plane of the surface at $P(x, y, z)$.

In case 3, the derivative of the space curve,

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t),$$

provides a linear relationship between the position differentials dx, dy and dz in the three-dimensional tangent space of the controlled variables and dt in the tangent line of the curve of the controlling variable:

$$\overline{dr} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \dfrac{df}{dt} \\ \dfrac{dg}{dt} \\ \dfrac{dh}{dt} \end{bmatrix} dt$$

Here [f', g', h'] describes the direction of the tangent line in the x-y-z space to the curve at the value of the controlling variable, t.

In case 4 the where several variables are controlled by several other variables, the multivariable derivative becomes a matrix (sometimes awkwardly called the Jacobian matrix and less frequently called the transformation derivative). This matrix linearly relates the differentials in the tangent space, T_R, at the image point in the range to the differentials in the tangent space, $T_{D,}$ of the domain.

Say transformation **T** is described by:

$$u = f(x, y, z)$$
$$v = g(x, y, z).$$

Then the relation between the differentials will be:

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad dv = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz$$

or $\quad \mathbf{dY} = \frac{dT}{dX}\, \mathbf{dX}$ where $\quad \mathbf{dY} = \begin{bmatrix} du \\ dv \end{bmatrix}, \qquad \mathbf{dX} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$

and

$$\frac{dT}{dX} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$

In summary, the connection between the spaces associated with a non-linear mapping **T** can be shown in the following diagram:

$$\begin{tikzcd} \textbf{Domain} \arrow[r, "T"] \arrow[d] & \textbf{Range} \arrow[d] \\ \mathbf{T_D} \arrow[r, "\frac{dT}{dx}"] & \mathbf{T_R} \end{tikzcd}$$

The domain is the, perhaps, non-linear space of the controlling variables. $\mathbf{T_D}$ represents the linear tangent space of the domain. The range is the, perhaps, non-linear space of the controlled variables. $\mathbf{T_R}$ represents the linear tangent space of the range. The transformation derivative $\frac{dT}{dX}$ linearly relates points in the linear space $\mathbf{T_D}$ to points in the linear space $\mathbf{T_R}$.

Features of the Transformation Derivative

Being a linear transformation, the derivative of a non-linear transformation provides all the features of linear mappings. While the derivative of a linear transformation is globally constant, not varying from point to point, the derivative of a nonlinear transformation does vary from point to point. The Jacobian determinant can vary from point to point, which implies that the ratio of differential volumes can vary. The Jacobian can be zero at special isolated, 'singular' points or along curves or surfaces etc. Strange effects occur at such points. Curves have cusps, surfaces fold back on themselves.

The number of differentials needed to describe the tangent space of the range cannot exceed the number of differentials needed to describe the tangent space of the domain. For ordinary, non-singular transformations the tangent space of the range has the same dimension as the tangent space of the domain. Only when the determinant of the transformation derivative, the Jacobian, is zero will the tangent space of the range have a smaller dimension than the dimension of the tangent space of the domain.

If a smooth non-linear transformation has a non-singular derivative:

- The range is locally stretched or shrunk continuously.
- The transformation derivative is invertible.
- The ratio of infinitesimal differential volumes, image/pre-image, equals the value of the Jacobian.

The places where the Jacobian determinant is zero are possible locations for cusps or other special features. Surfaces may branch along the curves where the Jacobian is zero.

Maximization in n-tuple Space

One-dimensional scalar spaces are linearly ordered. Either $a > b$, $a = b$ or $a < b$. Such is not the case when a space has a dimension equal to two or more. Examine the points on a tilted ellipse in a two-dimensional space. The point where the value of the horizontal coordinate, x is maximized is not the same as the point where the vertical coordinate, y or the point where the distance to the origin, r is maximized. Every point (x, y) on a circle, concentric with the origin has the same radius. Vectors cannot be ordered. Vector magnitudes, being scalars, can be compared but not the vectors. In this light we can use ordinary derivatives to maximize functions of a single variable and gradients to maximize functions of several variables but the idea of maximizing an n-tuple function of one or more variables is either unworkable or will take more careful consideration.

Applications of Multi-Variable Derivatives

Case 4, Several Variables Controlling Several Other Variables: The applications of the multi-variable derivatives stem from the derivative's features. The derivative can be used to compute the approximate effect on the controlled variables of small changes in the controlling variables. This line of thought opens the door to the study of Sensitivity Analysis. If the transformation derivative is invertible, the transformation derivative can be used to compute approximately the required deviation in the controlling variables needed to produce a given small change in the controlled variables. In situations where changes of coordinates are required the transformation derivatives must appear in the integrals for invariants such as arc length, surface area and volume.

Case 3, One Variable controlling Several Variables: When one variable controls several, changing the controlling variable produces a one-parameter curve in the space of the controlled variables. The derivative of the n-tuple of controlled variables with respect to the controlling variable determines the direction of the tangent line of this curve. If all the components of an n-tuple derivative are zero for some value of the controlling variable, then this point is a stationary point. No direction is evident. Maybe at this point the curve is changing direction discontinuously; the point is a cusp. The equations for the curve require further study at such a point. If the situation being studied is motion of a particle then the controlling variable is time and the controlled variables represent the position of the moving particle. The multivariable derivative will describe the velocity and the second derivative will describe the acceleration.

Case 2, Functions of Several Variables: All functions of several variables with a non-zero or a non-infinite gradient have a unique direction in the domain of maximum rate of change. The gradient of such a function will determine this direction. A direction of minimum rate of change lies directly opposite to the direction of maximum rate of change. Incrementally following the path of minimum rate of change is the germ of an idea for finding a minimum for a function of several variables. This idea is called the method of steepest descent.

All functions of several variables with a non-zero or a non-infinite gradient have directions in the domain that will maintain a constant value for the controlled variable. The gradient of a function of several variables is perpendicular to this direction or directions. For a function of two variables these directions are tangent to the level curves of the surface.

The gradient of a function of several variables can be used to determine the rate of change of the controlled variable in any specified direction in the domain.

If all the components of the gradient of a function of several variables are zero, then small changes in the controlling variables will have miniscule effects on the controlled variable. Such points where the gradient is zero, are locally stationary and may be extrema or saddle points.

Summary

This paper is a tour of the most important aspects of differential multivariable calculus. Curves in n-dimensional space are described by n equations in n + 1 variables and therefore have one degree of freedom. Surfaces in n-dimensional space are described by n-2 equations in n variables and therefore have 2 degrees of freedom. Linear spaces are flat that is, not curved or warped and contain the origin. Linear co-ordinatizations are straight, parallel and uniformly spaced. Essentially, linear spaces are described by linear (that is 1^{st} degree, no constant term) equations. The tangent line to a curve at a point is a straight line with the same direction as the curve at that point. The tangent plane to a surface at a point is a plane containing the point with the same normal direction as the surface.

Linear Algebra in its essence is the study of an algebra not a metric space. The vectors of linear algebra allow for addition and scalar multiplication but do not have properties of either length or angle. The inclusion of the axioms relating to dot products permits vectors to have the properties of lengths and angles. Linear Algebra with a dot product provides a wonderful tool to study the flat linear spaces and transformations of multivariable calculus. Linear Algebra with its concepts of independence, spanning, dimension and null spaces should be a prerequisite to the study of multivariable calculus. Conversely, it is the understanding of non-linearities that provides the insight into what is special about linear spaces and their transformations.

The idea of describing functions as models of control was introduced. Intelligent students wonder why they are studying functions and calculus. The standard currently available answers do not appear satisfactory. Students may find the concept of control as suitable to both the subject and to their needs. If students recognize the reality of functions as objects worthy of study then faculty can dispense with the introduction of distracting applications as is currently fashionable in the mathematics teaching community.

Four categories of finite-dimensional multivariable functions were highlighted, depending on the number of equations and the number of variables. A definition of the derivative that applies to each of the categories and that postpones or avoids the machinery of deltas and epsilons was introduced. This definition extends to the calculus of variations and to functional analysis. How the definition applies to each of the categories was described.

Maximization is an important topic in calculus and its extension into multi-variable calculus is equally as important. The gradient is clearly seen to be extension of the ordinary derivative to functions of several variables and just like the ordinary derivative must be zero at a stationary point. Applications of the other multi-variable derivatives demonstrate that the multivariable derivatives behave naturally as extensions of the derivative concept to multivariable control.

Twisted curves, warped surfaces, tangent lines and planes, degrees of freedom, multivariable control, multivariable derivatives, maximization, all are studied, stripped of the irrelevancies required by other disciplines, in multivariable calculus. We, mathematics teachers are the declared custodians of these wonderful concepts, discoveries, inventions, ideas and techniques. We must do more to ensure these ideas are presented clearly and made more accessible to our students.

Reference:
1. Struick, Dirk J. "Lectures on Classical Differential Geometry" Addison-Wesley Publishing Company, Inc. Reading, Massachusetts 1961

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Throughout his career Dr. Grossfield, has combined an interest in engineering and mathematics. He earned a BSEE at the City College of New York. During the early sixties, he obtained an M.S. degree in mathematics at NYU at night while working full time as an engineer for aerospace/avionics companies. He studied continuum mechanics in a doctoral program at the University of Arizona. He is licensed in New York as a professional engineer and is a member of ASEE, IEEE, SIAM and MAA.