WHAT IS COLLEGE ALGEBRA?

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Preface

In every well-planned course, only one thing is studied.

In arithmetic, numbers are studied. After studying arithmetic one should know the various kinds, forms, operations, properties of and relations between numbers. A student should end up feeling confidently when working with numbers.

In geometry, shapes are studied. After studying geometry one should know the various kinds, operations, properties and relations of shapes. A student should end up feeling confident when working with at least polygonal shapes and circles.

So, what are the objects of study in College Algebra and Calculus?

Functions

The major concerns in every quantitative scientific area are the relationships between the variables of that area. Since in the classical quantitative areas these relationships were continuous and smooth, the mathematicians of yesteryear developed the concept of the function and the techniques of using functions. The concept has proven to be a success and deserves a place of honor in the study of mathematics along with numbers and shapes. Technical competency in our modern world requires a thorough understanding of functions. College Algebra and Calculus comprise a study of a particular class of relationships between variables, in particular, continuous, smooth functions.

Suppose we are interested in hurricanes. We might focus our attention on the variables: maximum wind speed, air pressure in the eye, cloud height and height of the storm surge. We might wonder what would be the effect of increasing wind speed. Would air preeure in the eye be increased? Would cloud height be increased? and storm surge? How can we think about or describe such relationships? The invention of the concept of a function provides a systematic way to study numerical relationships. Functions lie at heart of quantitative design and planning. What, then, are **functions** and what is there to study about them?

What are Functions?

A study of a simple quantitative system might be started by identifying the variables that describe the system. It is possible that there may be no relationships between these variables. However in special cases the values of some variables determine the values of other variables. When one or more variables determine the value or values of another variable the relationship between the variables is called a function. The relationship might describe variables that happen to move together, indicating co-variation. If the independent variable is time, the relationship describes the evolution of the dependent variable, a trend. Often the relationship describes control. Moving the independent variable forces the variation or motion of the dependent variable. These relationships, continuous functions, are the focus of Algebra and Calculus courses. Continuous functions are needed to describe co-variant variables, trends, and control.

Descartes invented the rectangular coordinate system which enabled the trends contained in a discrete table of values for a function to be displayed as a set of dots or a connected set of line segments. The two dimensional coordinate system enabled an equation in two variables to be graphed as a continuous curve. The use of tables, curves and equations to represent continuous functions has become conventional in modern science. Modern scientists, engineers and technicians must be able to recognize and use functional relationships, not only for their personal view of the deterministic continuous world, but also to be able to communicate with their colleagues and associates about that world.

Functions are special relationships between variables. In the study of some physical systems, the value of one variable determines the value of another variable. However, most of the time, variables are not related. The speed of passing car has no connection to the temperature inside a refrigerator. The length of a rectangle does not determine the area of the rectangle. These are unrelated variables. The area of the rectangle is not a function of its length. The refrigerator temperature is not a function of the car’s speed. However, the radius of a circle, indeed, does determine the area of the circle. This connection is special. The area of a circle is a function of its radius. Functions are related (or connected or linked) variables.

When one variable, say x determines or controls the value of another variable, say y, we say that there exists a functional relationship between x and y. If we call this relationship f, than it has become customary to write $y = f(x)$. This means that if we pick or know a value for x then, because of the functional relationship f, we can determine y. The controlling variable (in this case x) is called independent and the controlled variable (y) is called dependent. If the function is known, it can be displayed by a table, a graph, and sometimes an equation.

Suppose we are studying circles. We might focus on the four variables: radius, (R), diameter, (D), circumference, (C) and area, (A). We know from geometry that the diameter, the circumference and the area are uniquely determined once the radius is known. These relationships are described by the equations:

$$D = 2 * R,$$
$$C = 2 \pi R \qquad \text{and}$$
$$A = \pi R^2 .$$

These equations describe in symbols the exact way in which D, C and A are related to R. One is free to choose any positive number for the radius, but once the radius is selected there is absolutely no freedom at all for the values of the diameter, the circumference or the area. The relationships describing this system provide examples of functions that would be symbolically described as:

$$D = f(R), \qquad C = g(R) \qquad \text{and } A = h(R).$$

In this system of four variables and three equations, R has been chosen to be independent. The remaining variables; D, C, and A that are determined by the equations are dependent.

Sometimes one equation in one variable will determine the value of the variable. However, one equation in two variables will not determine the values of the variables. To determine the values of two variables, two equations are required. Loosely, **n equations are required to determine n variables**. If we had 4 equations with 6 variables we might choose 2 variables as independent. Now the 4 equations could be used to determine the remaining 4 remaining dependent variables. Let us examine the simple case of one equation in two variables. If one of the two variables is chosen as independent, the equation can be used to determine the value of the other variable. These equations describe the functions that are studied in Algebra and Calculus. Instead of describing functions with equations, functions may be described using a numerical table format. A table format may be convenient to use if the table is not too long and the desired values of the independent variable are listed. A table description of the function $A = \pi R^2$ is shown in Table 1.

R	$A = \pi R^2$
0	0
1	$A = \pi = 3.14159$
2	$A = 4\pi = 12.56637$
3	$A = 9\pi = 28.27433$
4	$A = 16\pi = 50.26548$
5	$A = 25\pi = 78.53982$
6	$A = 36\pi = 113.09734$

Table 1

Another format for describing functions is called a graph. In a graphical format the independent variable is plotted on a horizontal scale while the dependent variable is

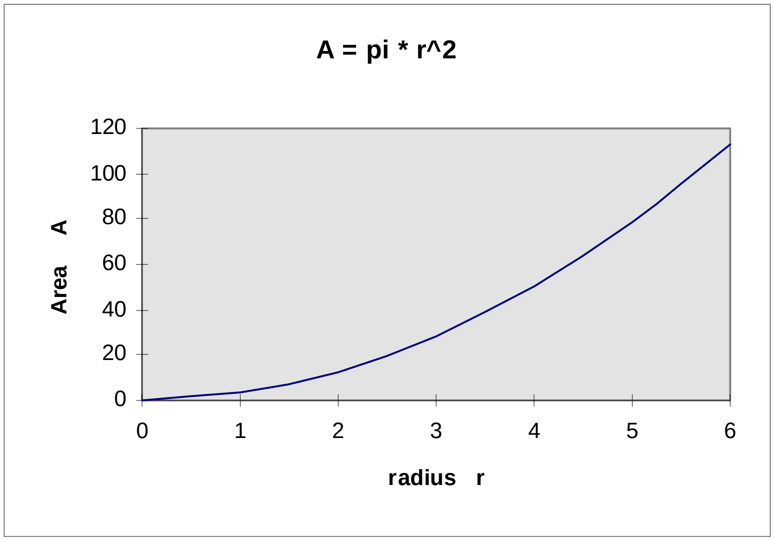

Figure 1

plotted vertically. In this format every equation in two variables appears as a curve. While a graphical format may lack precision, the format is invaluable for depicting trends and many of the features of functions. The properties of increasing, decreasing, maxima, minima, zero crossings and concavity become obvious in a graph. The graph for the function $A = \pi R^2$ is shown in figure 1.

All three formats,--- equations, graphs and tables have advantages and disadvantages but they are commonly used to describe quantitative relationships and a beginning algebra student must develop the ability to use any form and conceive of the problems that may occur in changing between the forms.

A curve that represents a function, where there is only one dependent variable value for every value of the independent variable, may cross a vertical line only once. If the curve has no gaps or jumps, the function is called continuous. If a continuous curve has no corners, the function describing the curve is called smooth or differentiable.

Properties of Functions

A list of the function properties treated in algebra and calculus is provided in Table 3. These properties all have graphical interpretations. Increasing means that as the independent variable increases, the dependent variable also increases; that is the points on the curve drift from lower left toward upper right. Decreasing means that as the independent variable increases, the dependent variable decreases; that is the points on the curve drift from upper left toward lower right. Functions that are increasing or decreasing everywhere are called monotonic. Maxima are the peaks of the dependent variable and minima are the valleys of the dependent variable. Functions describing curves that have more than one value of the dependent variable for a given value of the independent variable are called multiple-valued functions.

Kinds of Functions

The functions of algebra and calculus, called by mathematicians, the elementary functions can be grouped, by their algebraic form or their method of construction into five categories: 1) Polynomials, 2)Rational functions, 3)Algebraic functions, 4) Transcendental functions and 5) Piecewise-defined functions. Typical functions of each category are shown in figures 2 through 6.

Following are verbal descriptions of the graphs of each of the kinds of functions:

1) **Polynomials** are gentle, single valued, continuous, smooth curves, defined for all values of the independent variable (say x) which start on the left at either + or - infinity and end on the right at either + or - infinity.

Polynomials have no point gaps, jumps, excluded intervals, cusps, multiple values, horizontal or vertical asymptotes (poles) or linear asymptotes for degree greater than 1, or periodicities. If you add, subtract or multiply polynomials, a polynomial results. Dividing polynomials produces a rational function.

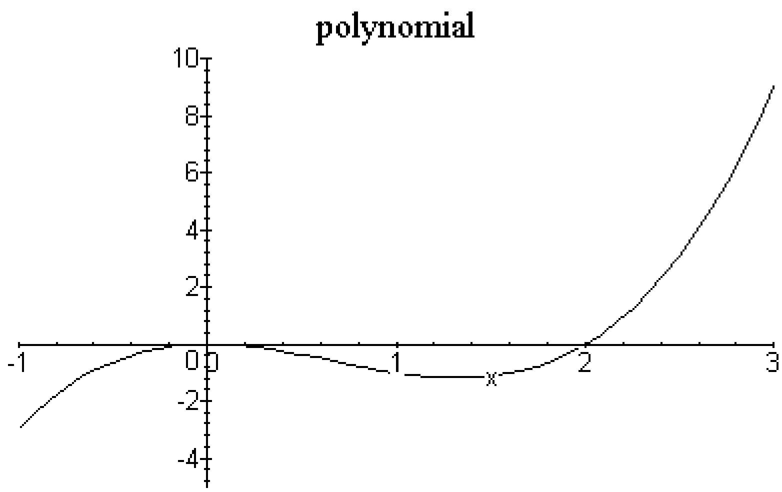

Figure 2 The polynomial $y = x^2 (x - 2)$

2) **Rational functions** are also single valued, smooth continuous curves defined for all x, except at a finite number of points where there may exist point gaps or vertical asymptotes. These curves behave at x = + and - infinity like polynomials but, perhaps, may have a horizontal asymptote.

Rational functions have no finite jumps, excluded intervals, cusps, loops, multiple values, or periodicities.

If you add, subtract, multiply or divide rational functions, a rational function is produced.

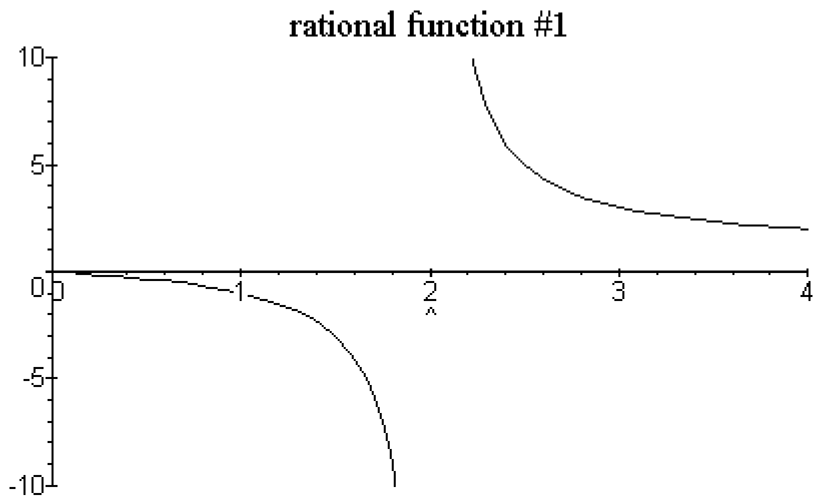

Figure 3 The rational function $y = 1/(x - 2)$

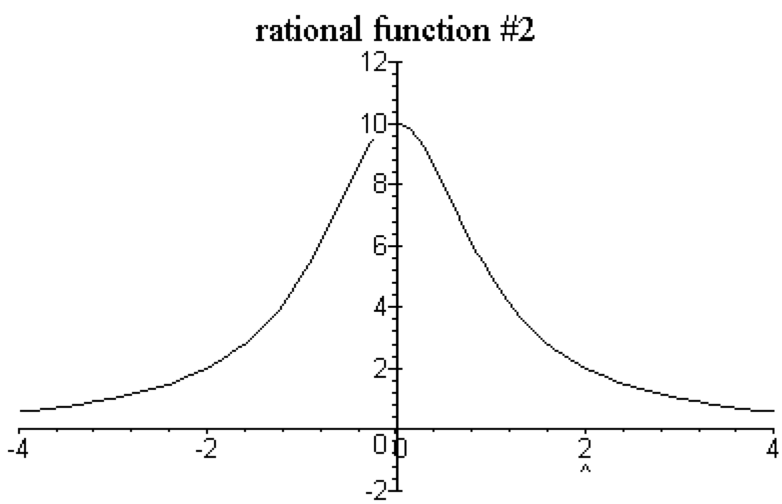

Figure 4 The rational function $y = 1/(1 + x^2)$

3) **Algebraic functions** are generally smooth continuous curves. These curves may be multiple-valued, may have cusps, loops, self intersections and may have excluded intervals and may be bounded or unbounded vertically and horizontally. A circle is a simple example of an algebraic curve.

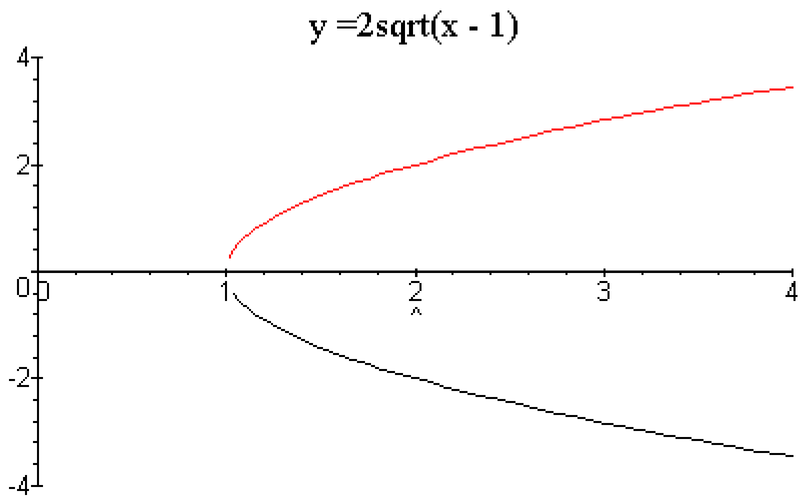

Figure 5 An algebraic function

Algebraic functions can not be periodic.

The above three categories of curves can intersect a non-coincident straight line only, at most, at a finite number (called the degree of the curve) of points.

4) **Transcendental functions** have few restrictions. They may be multiple valued, periodic and may intersect a straight line an infinite number of times. They may have an infinite number of asymptotes or may spiral.

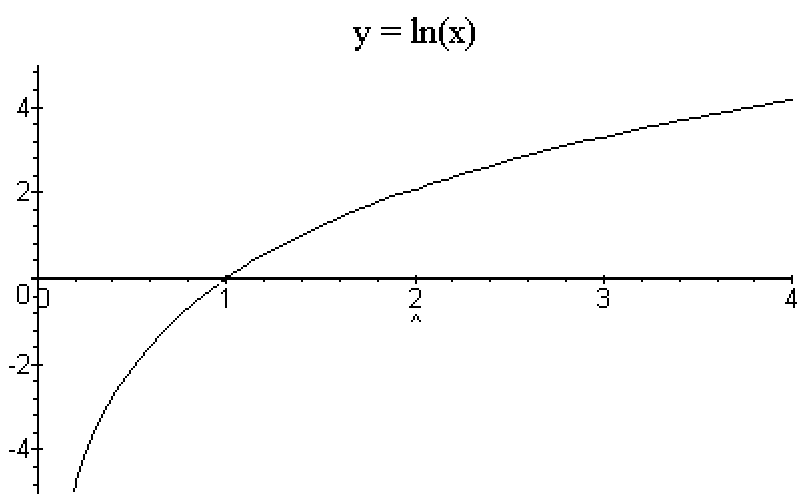

Figure 6 A Transcendental Function $y = \ln(x)$

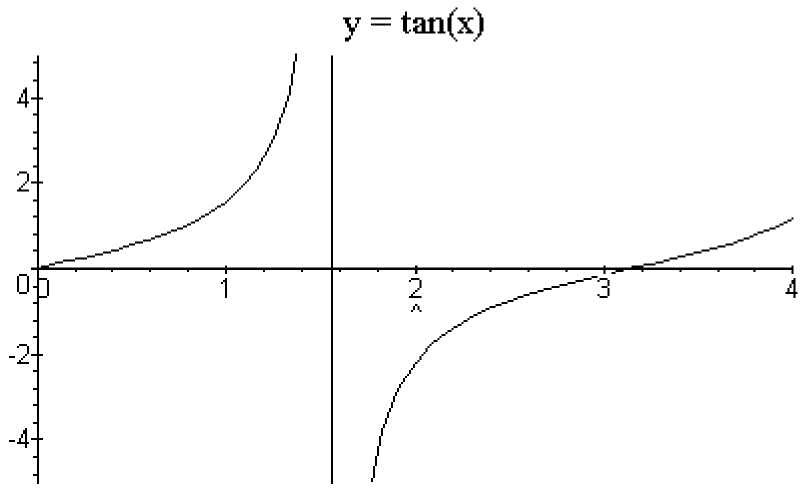

Figure 7 A Periodic Transcendental Function y = tan(x)

5) **Piece-wise defined functions** are made by piecing together segments of the above categories and include the step, staircase, absolute value and sawtooth. These may take on any of the properties of the curves of which they are made and may have jumps.

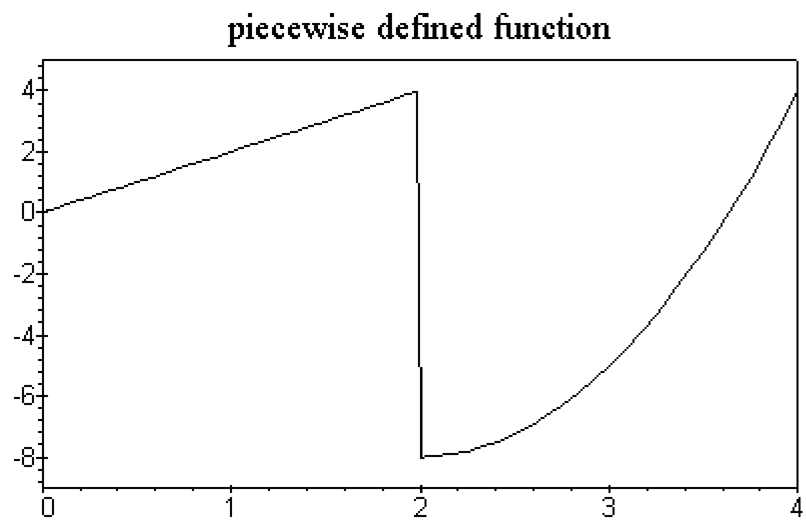

Figure 8

Operations on Functions

Many functions of algebra and calculus are obtained by performing operations on more elementary functions. The operations can be classified as unary or binary according to whether the operations require one or two operands (note: the operands are functions).

Binary operations, such as addition, subtraction, multiplication and division, take two functions as operands and produce a third function. Unary operations, such as negation, reciprocation or squaring, take one function as an operand and produce a second function. The elementary operations (that is, all the following operations except those on the line marked *) are obtained by performing arithmetic operations on the dependent variable for each value of the independent variable. The operations on the line marked * called the derivative and the integral are studied in detail in a calculus course. By the end of a calculus course, a student should be familiar with all the operations below:

unary operations: af(x) , -f(x) , $\frac{1}{f(x)}$, $(f(x))^2$, $(f(x))^n$, $\sqrt[n]{f(x)}$

* $\frac{df(x)}{dx}$, and $\int f(x)dx$,

binary operations: + , - , * , and / .

A student should seek a graphical interpretation for each of the above operations. Following are some important theorems concerning operations and the preservation of the properties of continuity, smoothness and monotonicity:

- Functions that result from adding, subtracting, or multiplying continuous functions will be continuous.
- Functions that result from adding, subtracting, or multiplying smooth functions will be smooth.
- Functions that result from adding, monotonically increasing functions will be monotonically increasing.
- Functions that result from dividing continuous functions will be continuous except where zeros occur in the denominator.
- Functions that result from dividing smooth functions will be smooth except where zeros occur in the denominator.

Objectives

Functions have major importance in every quantitative area of study and the study of functions will significantly impact on a student's understanding of these areas. Some may claim that the worth of algebra and calculus may best be seen by examining the many areas where calculus can be applied, that is where functions are used. I. don't think so. I believe that a student who understands what a function is and who understands the basic principles of the area that he is studying will have no trouble seeing if and how calculus will apply to his endeavor. However, below in Table 4, is listed a number of clear applications of calculus to the study of functions and to the geometry of curves.

Symbolic Forms of Functions

Just as numbers have many forms that are best suited to the application of the moment, functions require a form to be described. Tables and graphs are commonly used to describe functions and there are several symbolic forms for describing functions that are listed below in Table 5. A large part of the traditional College Algebra course has been concerned with the study of algebraic forms, in particular, the techniques of changing from one form to another. A function such as:

$$y = x^2 - 4$$

can also be described as:

$$y =(x - 2)(x + 2).$$

For every value of x, squaring and subtracting 4 produces the exact same value as multiplying the sum and difference of x and 2. Both equations will produce the same tables; both will produce the same graphs; yet the forms of the equations appear different and the operations required to compute y, when x is known, are not the same. Each form has advantages and disadvantages. What a strange and interesting possibility.

Summary

During the preceding discussion, I have tried to convey the important ideas surrounding the concept of mathematical function, in particular, the kinds, properties, operations and forms of functions. This quick tour was meant to introduce you to the lay of the land rather than to provide explanations. I am sure that this tour must have raised some questions in your mind regarding the above topics and I hope that during the remainder of the semester you will seek the explanations, details and answers that should comprise a course in Algebra.

References: 1. Grossfield, Andrew "On the Classification of Functions and Curve Plotting" Proceedings of the 1990 ASEE Annual Conference, Session 2665 (1782-1784)

2. Grossfield, Andrew "On the Intrinsic Structure of Calculus" Proceedings of the 1995 ASEE Annual Conference, Session 1265 (311-315)

Kinds of Functions

polynomials
rational functions
algebraic functions
transcendental functions
piece wise defined functions

Table 2

Properties of Functions and Curves

(Distinguish point properties
from region properties)

1. intercepts,
2. symmetries, even, odd, periodic, etc.
3. excluded intervals,
4. bounds,
5. asymptotes,
6. multi-valued,
7. continuity,
8. discontinuity
   - isolated discontinuities
     - finite jumps
     - infinite jumps, poles
     - gaps
     - oscillatory discontinuities
9. differentiability,
10. slopes,
11. monotonicity,
12. extrema,
13. concavity,
14. points of inflection,
15. cusps,
16. radii of curvature,
17. arc length,
18. area
19. connected
20. closed,

Table 3

Objectives of Calculus

Evaluation:
* evaluating functions of a single variable
* graphing functions
* finding roots
* finding intersections
* evaluating inverse functions

Differentiation:
* finding extrema and points of inflection
* finding tangent lines
* describing direction
* describing rate of change

Integration:
* finding areas of regions with curved boundaries
* finding arc length of curves
* finding surface area and volumes of revolution

Table 4

Symbolic Forms of Functions

direct		y = f(x)
inverse		x = g(y)
implicit		f(x, y) = 0
composite (chain)		y = u(w); w = v(x)
parametric		$y = g(t)$ $x = f(t)$
polar		r = f(θ)
series	Taylor Fourier	

Table 5

Algebra/Calculus Topic Tree Diagram

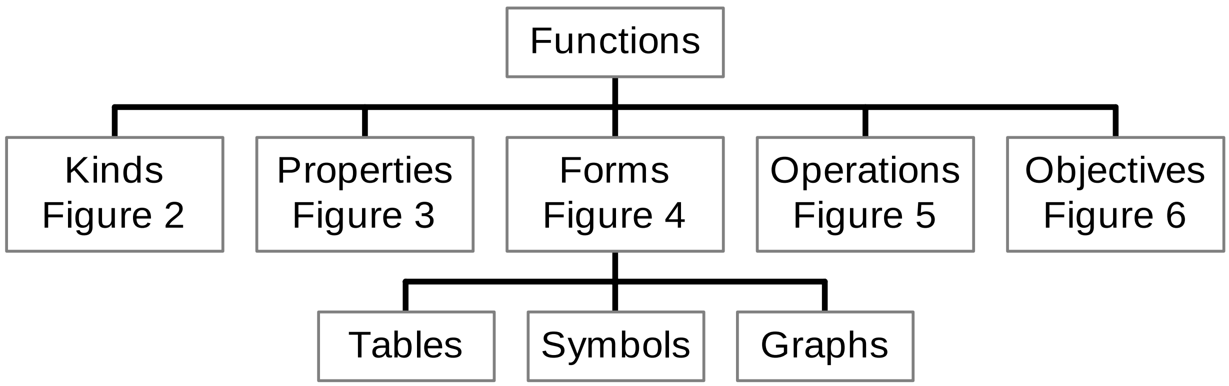

Figure 9